Technology as a support for generating and presenting proofs in geometry

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Techology and proving in school geometry?

Automated theorem provers (TPS approach)

Tools for helping studens to understand, learn and build deductive proofs (non-TPS approach)

Both levels are important (Sfard, Dubinsky)

What topics ?

What tools?

What didactic approach?

Technology and non TPS proving

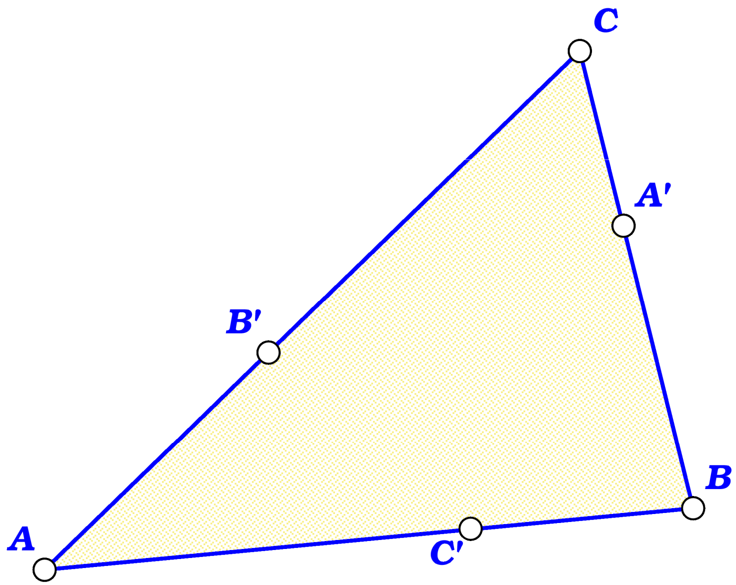
- **Observation of facts**
- **Building/writing up proofs**
- **Validating proofs**
- **Presenting proofs**
- **Learning to prove facts**

OK Geometry

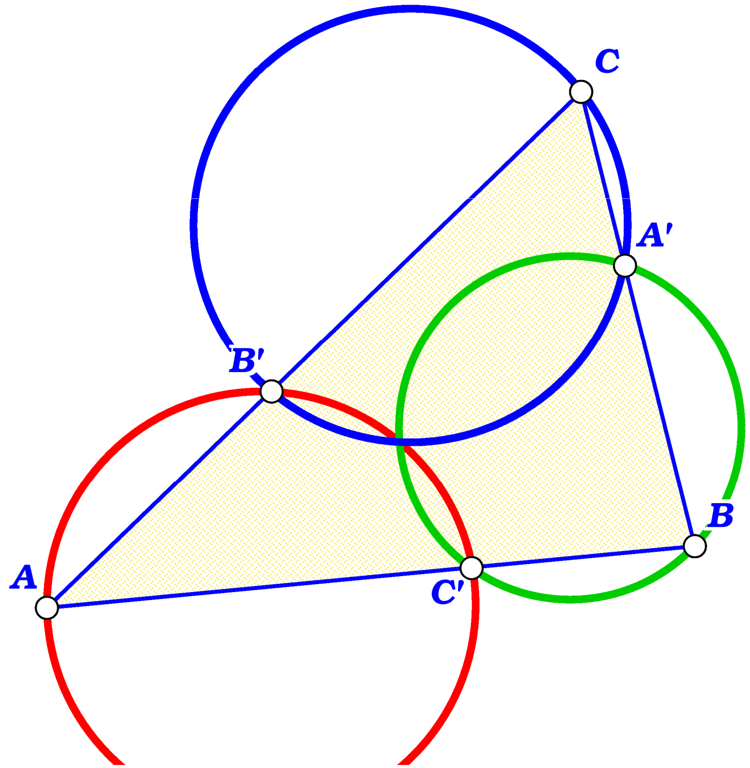
- **OK Geometry** is an **observational tool**. It detects (highly probable) properties of dynamic constructions (eg. from GeoGebra, Cinderella).
- With **OK Geometry** one can **generate hypothesis** and **proof-oriented exercises**.
- **OK Geometry** does not prove theorems, but may help students in building up proofs. Students need to relate the observations and provide the arguments.
- **OK Geometry** is free.
  See http://z-maga.si/index?action=article&id=40

Basic approach of OK Geometry

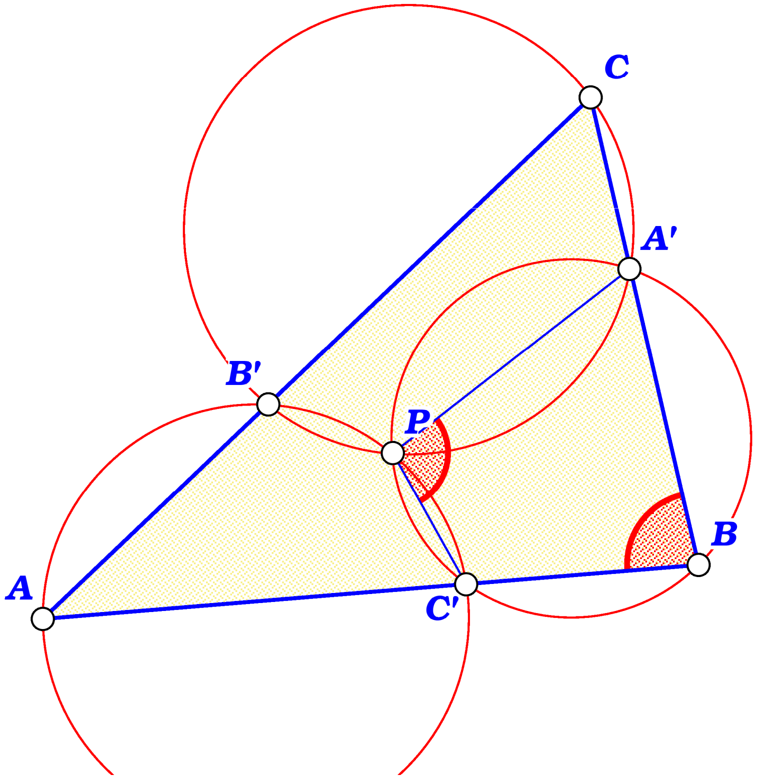
On each side of a triangle there is a point. Observations?

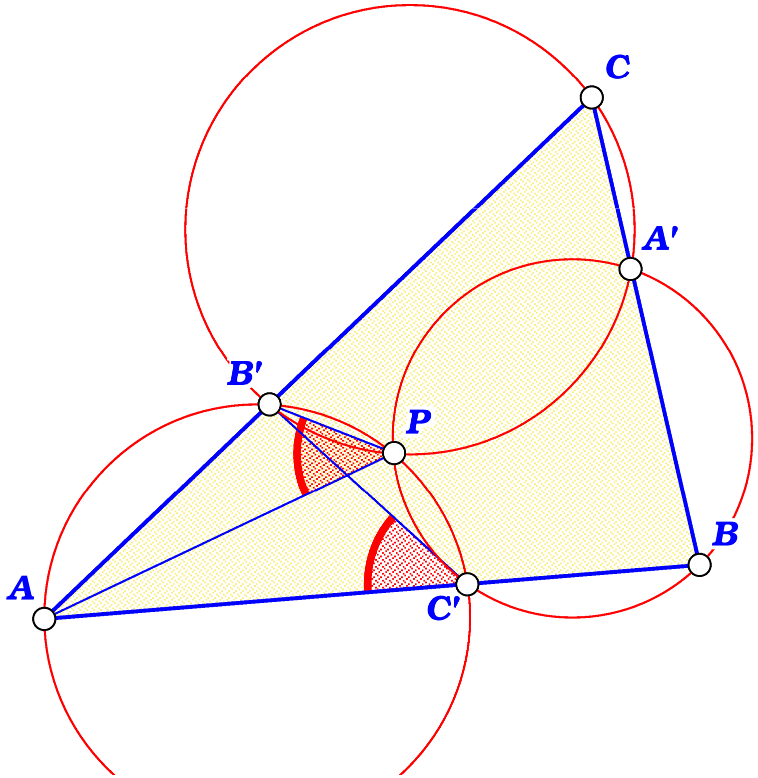


An OK observation: The three circles meet in a common point.

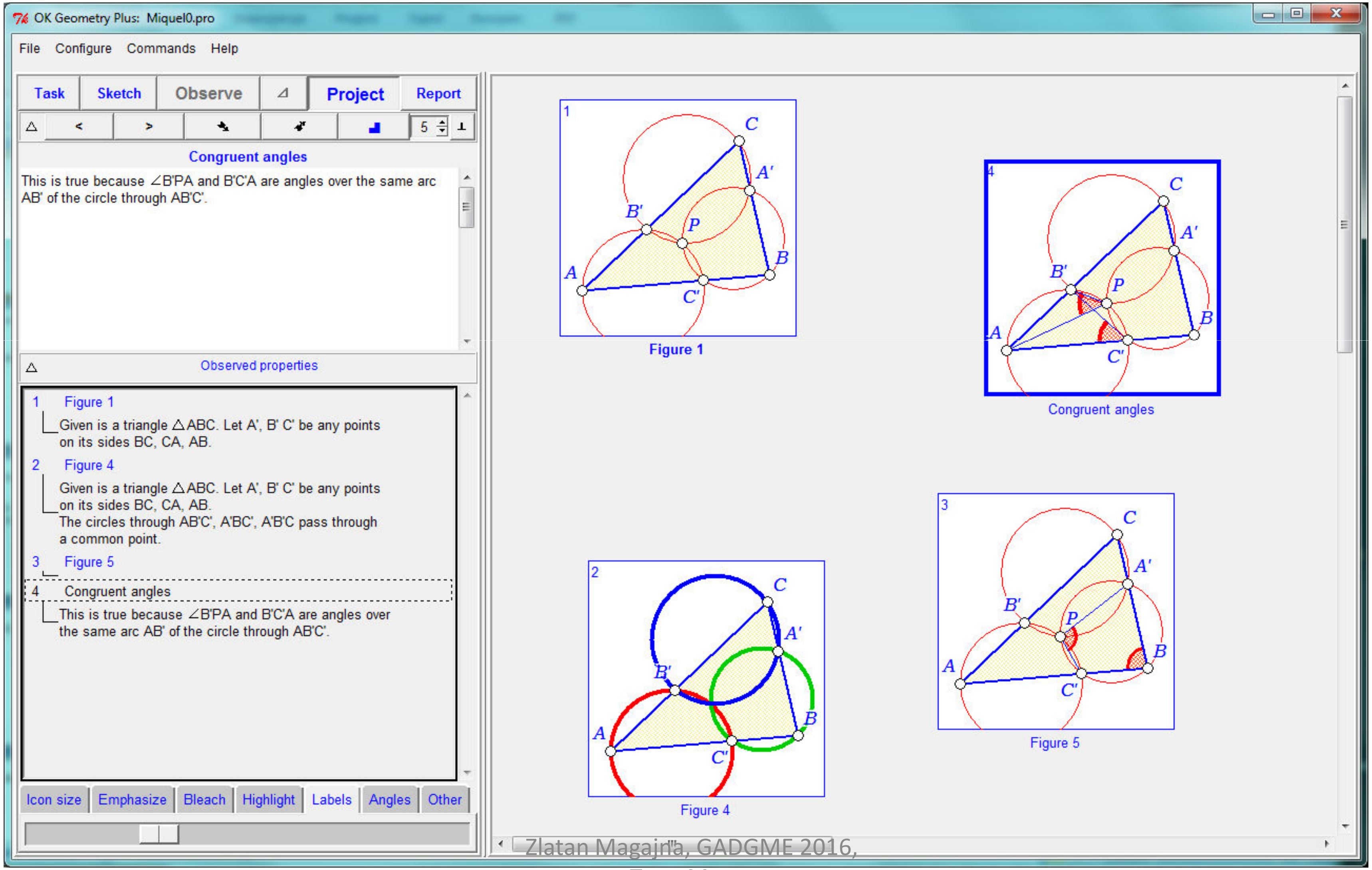


One selects observations that may possibly be of use in proving the three circles claim.

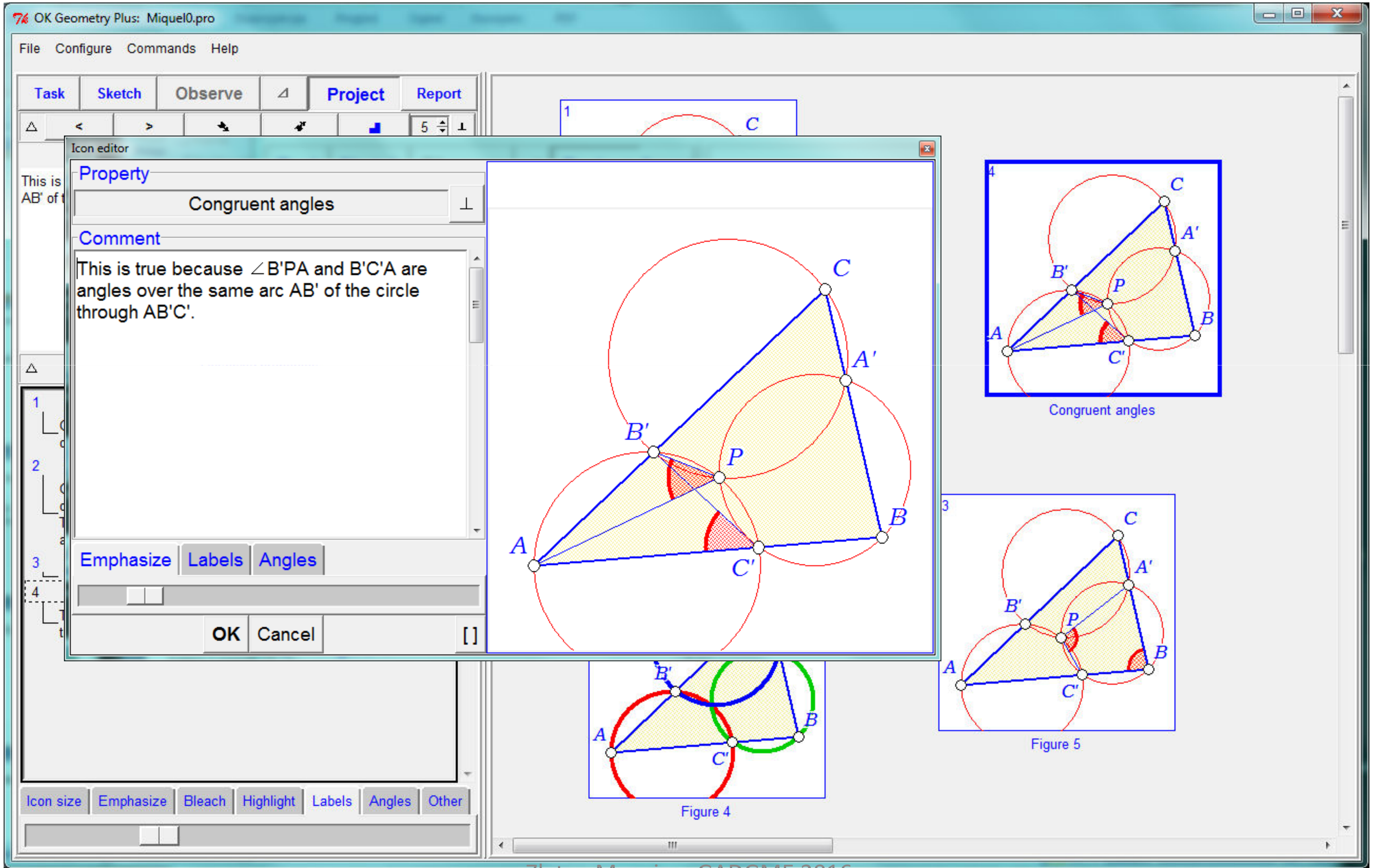




The selected properties are collected and organised into a proof ….



… and deductive argumentations are provided

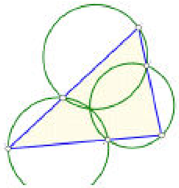


The obtained proof in paragraph form

OK Geometry Report : Miquel theorem; ; (Miquel_hyper_proof.pro); 2016.8.25

Miquel theorem

1 Miquel theorem



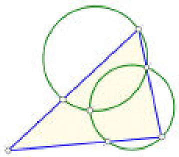
Given is a triangle ΔABC. Let A', B', and C' be arbitrary points on the sides BC, CA, and AB.

The three circles through A,B',C', through A',B,C', and through A',B',C allways meet in a common point.

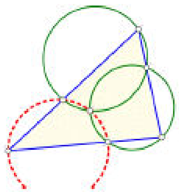
Comment:

2 Strategy of the proof

Let P be the intersection other than A' of the circles through B,C',A' and through C,A',B'. We shall prove that P lays on the circle through A,B',C'.



3.0 Idea of the proof



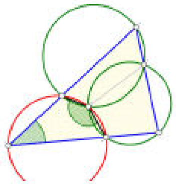
We shall prove that A, C', P, B' are cocyclic, ie. that AC'PB' is a cyclic quadrilateral. To prove this we shall use the theorem:

A quadrilateral is cyclic if and only if its non adjacent angles are supplementary.t

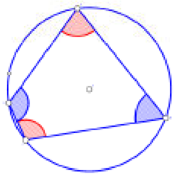
OK Geometry Report : Miquel theorem; ; (Miquel_hyper_proof.pro); 2016.8.25

3.1 Theorem

A quadrilateral ABCD is cyclic if and only if its non-adjacent angles are supplementary.



3.2 Proof ⟹



(⟹) Let ABCD be a cyclic quadrilateral. Thus ABCD is inscribed in a circle, let its centre be S.

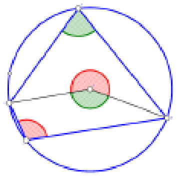
Consider the oppposite angles ∠A and ∠C. These angles are related to complemetary arcs BCD and DAB. The related complementary central angles ∠BSD and ∠DSB add up to 360°. Since the central angle of an arc is allways twice the related arc angle, the angles ∠A and ∠C add up to 180°.

3.3 Proof ⟸

(⟸) Assume nor thar in the quadrilateral ABCD the angles ∠A and ∠C are supplementary. We claim that ABCD is cyclic.

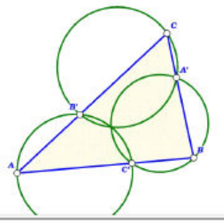
Consider the circle k through A, B, and D. Assume, by contradiction, that C does not lay on k, but, for example, inside it. Let C' be the intersection, other than D, of the line BC with the circle k. Since ABC'D is cyclic, the angle ∠C' is congruent to ∠DCB, since both are supplementary to ∠A. But this is impossible since in the triangle ΔC'CB the exterior angle at C cannot be congruent to the opposite interior angle.

An analogous argument holds in case C lays outside the circle k.



The obtained proof in two column form

OK Geometry Report : Miquel theorem; ; (Miquel_hyper_proof.pro); 2016.8.25



Miquel theorem

1 Miquel theorem

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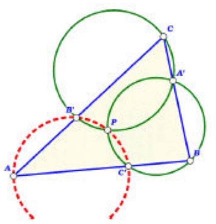
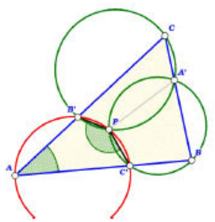
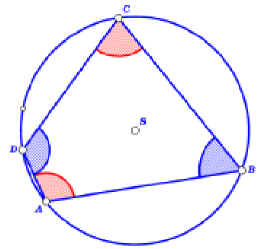
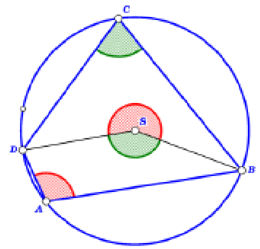
The three circles through A,B',C', through A',B,C', and through A',B',C allways meet in a common point.

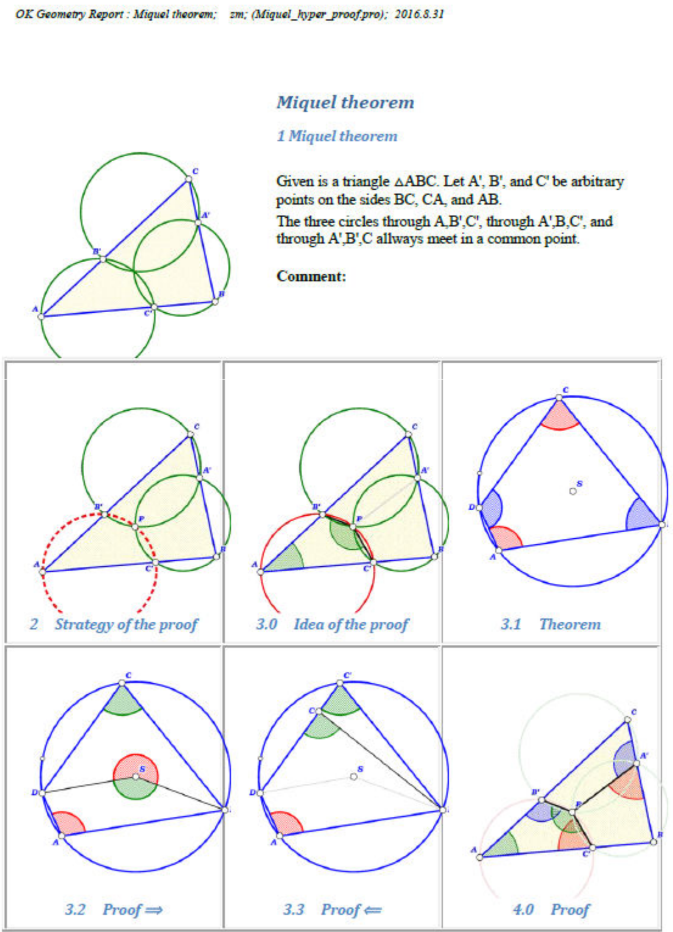
Comment:

	*2 Strategy of the proof*	Let P be the intersection other than A' of the circles through B,C',A' and through C,A',B'. We shall prove that P lays on the circle through A,B',C'.
	*3.0 Idea of the proof*	We shall prove that A, C', P, B' are cocyclic, ie. that AC'PB' is a cyclic quadrilateral. To prove this we shall use the theorem: A quadrilateral is cyclic if and only if its non adjacent angles are supplementary.t

OK Geometry Report : Miquel theorem; ; (Miquel_hyper_proof.pro); 2016.8.25

	*3.1 Theorem*	A quadrilateral ABCD is cyclic if and only if its non-adjacent angles are supplementary.
	*3.2 Proof ⟹*	(⟹) Let ABCD be a cyclic quadrilateral. Thus ABCD is inscribed in a circle, let its centre be S. Consider the oppposite angles ∠A and ∠C. These angles are related to complemetary arcs BCD and DAB. The related complementary central angles ∠BSD and ∠DSB add up to 360°. Since the central angle of an arc is allways twice the related arc angle, the angles ∠A and ∠C add up to 180°.

Other forms of proof presentation


OK Geometry Report : Miquel theorem;   zm; (Miquel_hyper_proof.pro); 2016.8.31

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Comment:

2 Strategy of the proof

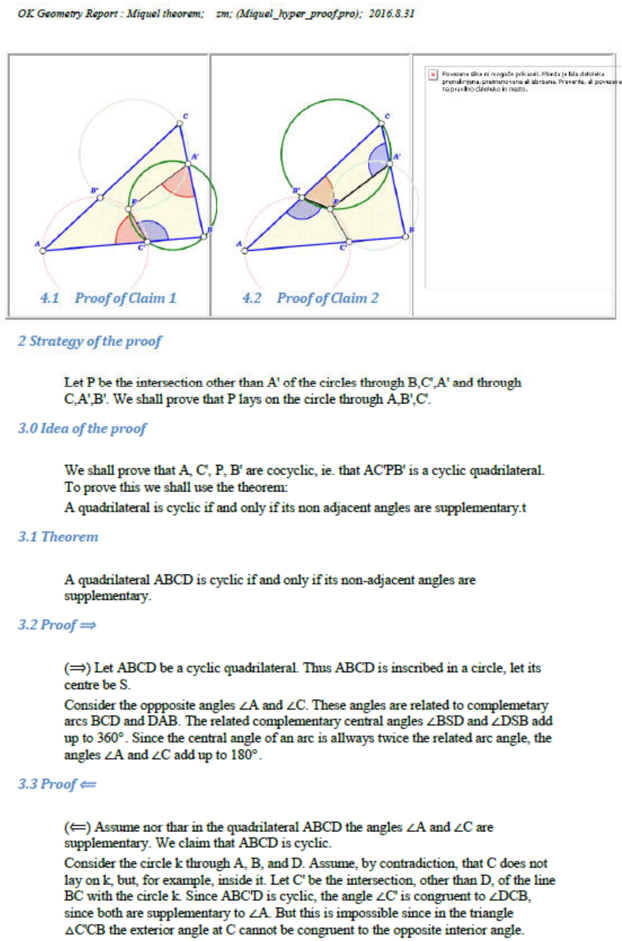
3.0 Idea of the proof

3.1 Theorem

3.2 Proof ⟹

3.3 Proof ⟸

4.0 Proof


OK Geometry Report : Miquel theorem;   zm; (Miquel_hyper_proof.pro); 2016.8.31

4.1 Proof of Claim 1

4.2 Proof of Claim 2

2 Strategy of the proof

Let P be the intersection other than A' of the circles through B,C',A' and through C,A',B'. We shall prove that P lays on the circle through A,B',C'.

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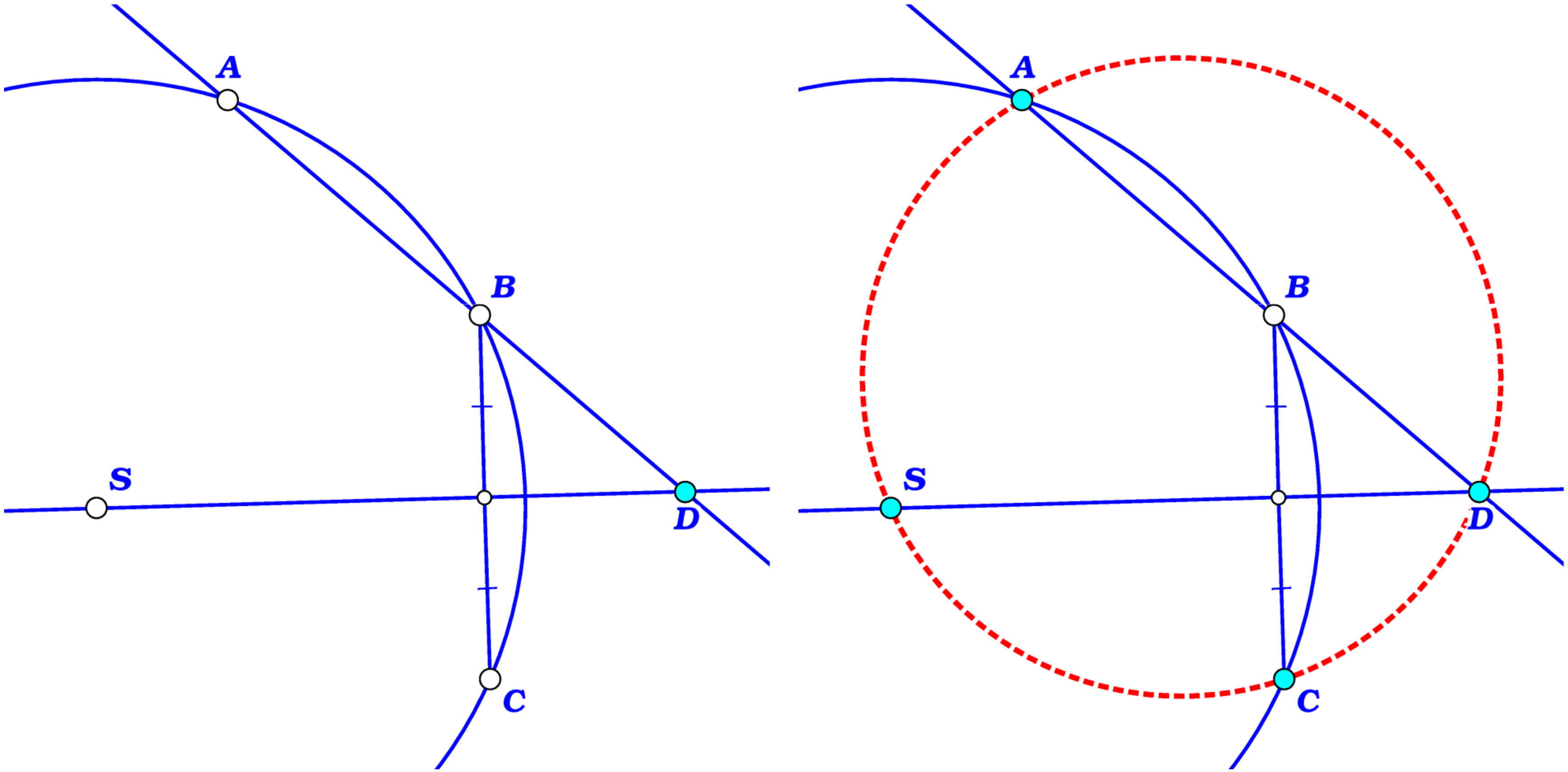
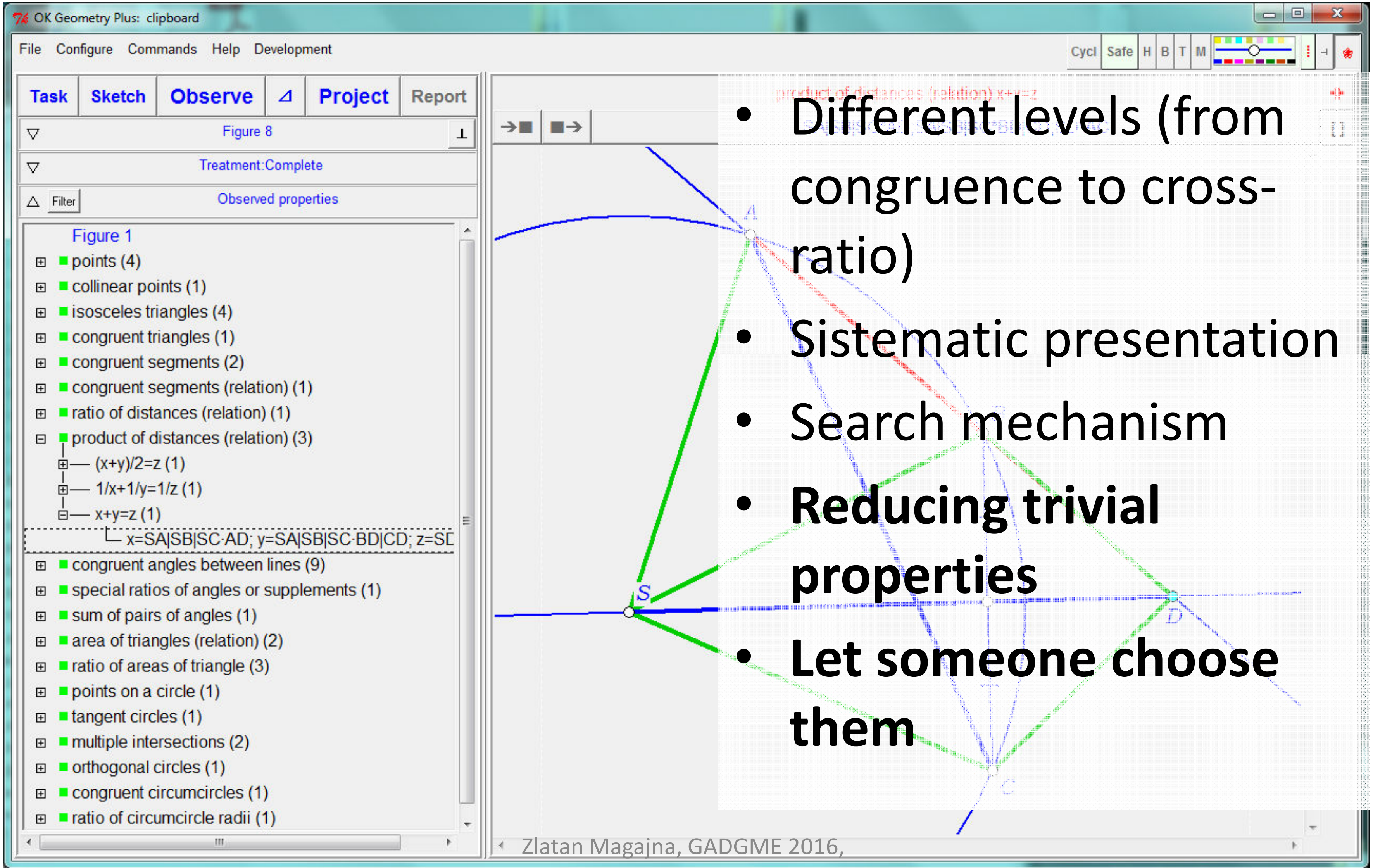
Problems in automated observation - Tons of detected properties



Menaging the observed properties

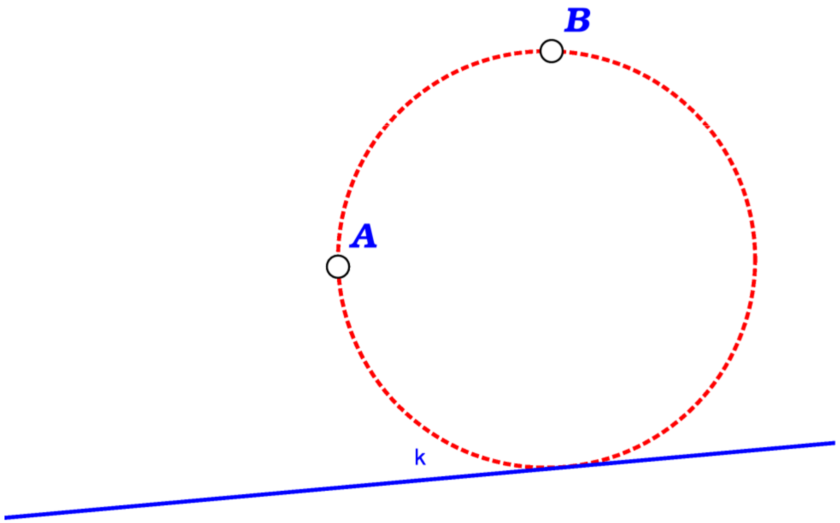


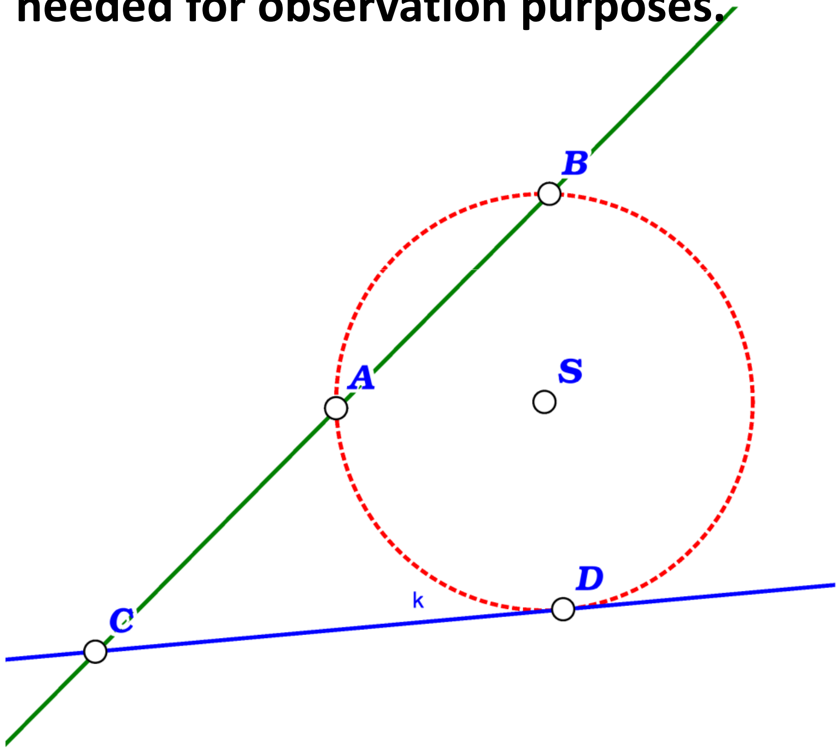
- Different levels (from congruence to cross-ratio)
- Sistematic presentation
- Search mechanism
- **Reducing trivial properties**
- **Let someone choose them**

Problems in observation – difficult objects

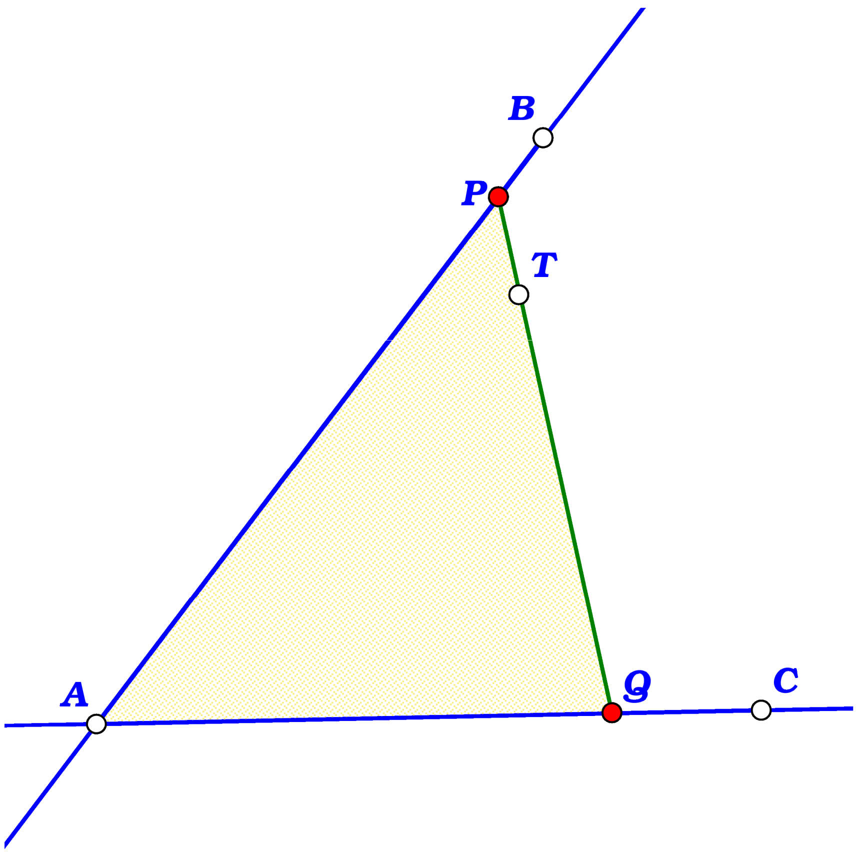
A circle passing through two given points and touching a given line.

Difficult objects are objects that one is unable to construct. Yet, the object is needed for observation purposes.





Difficult and new objects

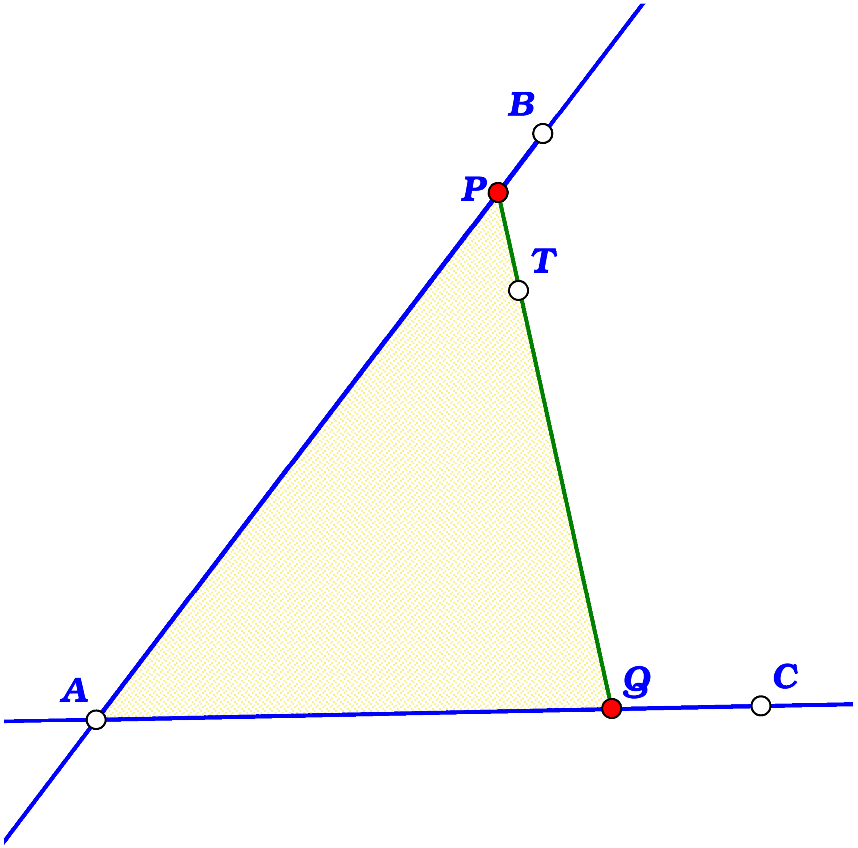


Given are two lines, AB and AC, and a point T. Find the shortest line segment PQ passing through T with P on AB and Q on AC.

PQ is a difficult object.

Difficult and new objects

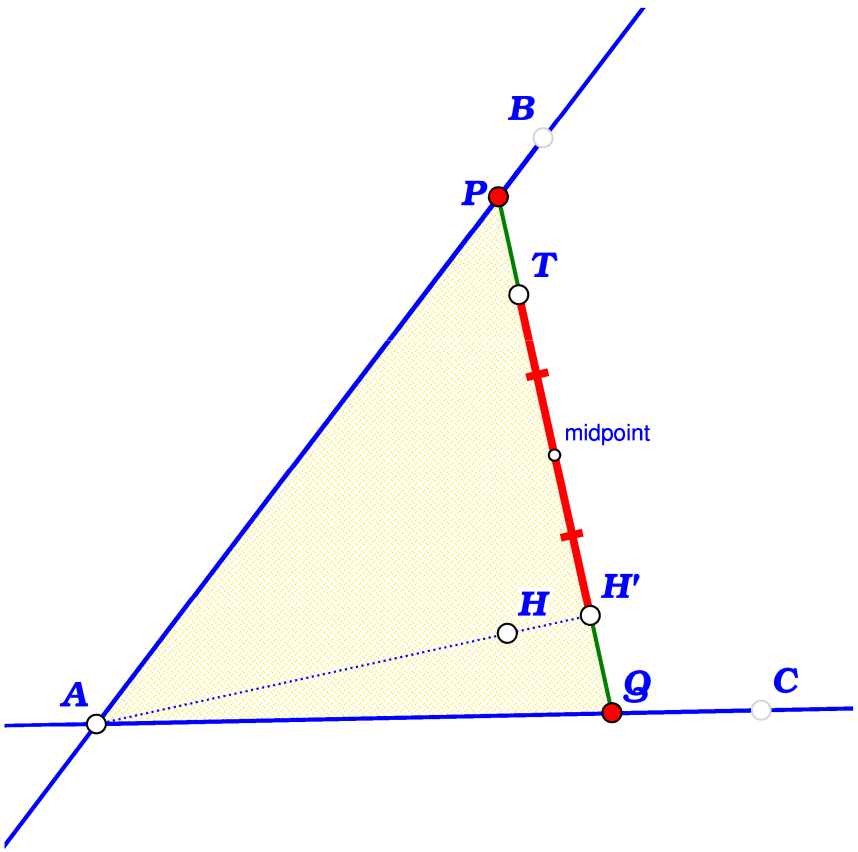
OK Geometry relates T to triangle ABC:



Consider the projection to the baseline containing T of these APQ related points: (35 items, great caution)

- X69: SYMMEDIAN POINT OF THE ANTICOMPLEMENTARY TRIANGLE (central projection from vertex to T)
- X4: ORTHOCENTER (central projection from vertex to mirror image of T wrp to side bisector)
- X25: HOMOTHETIC CENTER OF ORTHIC AND TANGENTIAL TRIANGLES (central projection from vertex to mirror image of T wrp to angle bisector)
- X20: DE LONGCHAMPS POINT (orthogonal projection to baseline to T)
- U043: 2nd ACACIA POINT (U43) (median parallel projection to midpoint of T and a triangle vertex)

Difficult and new objects



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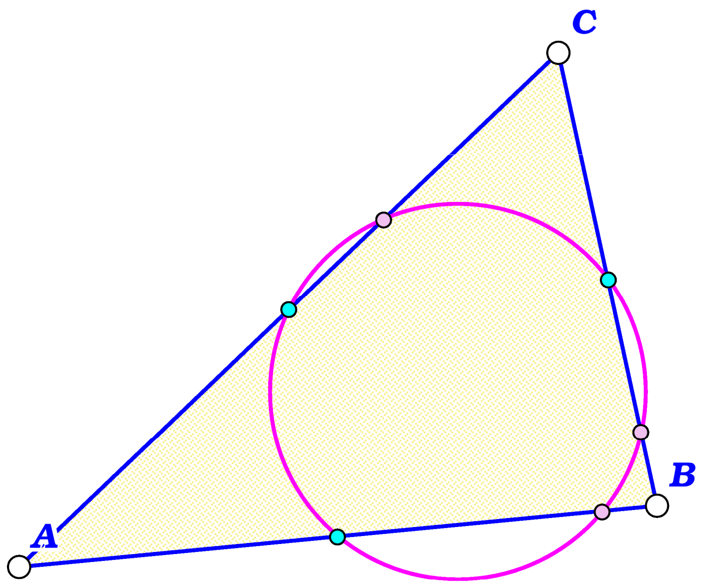
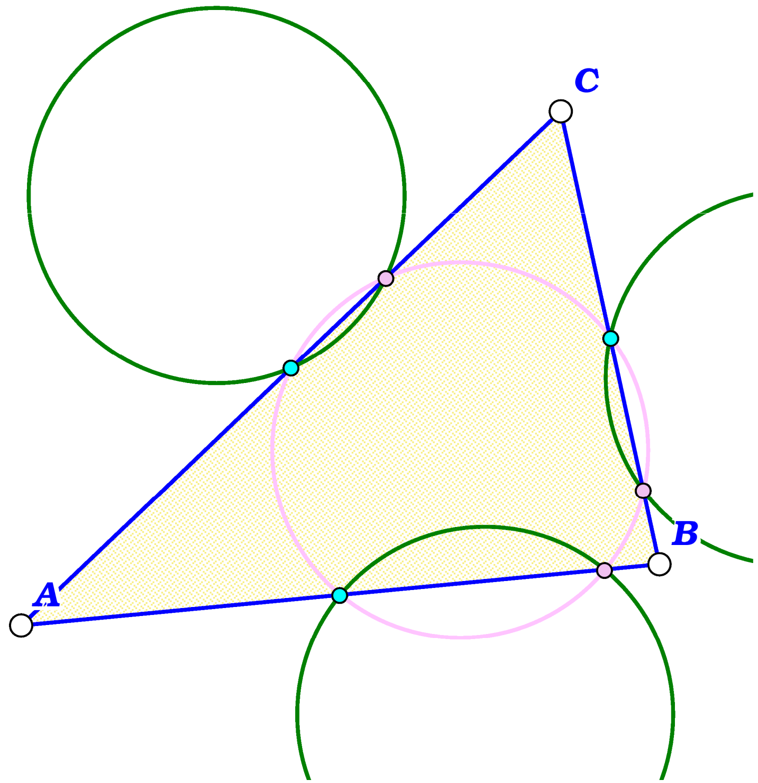
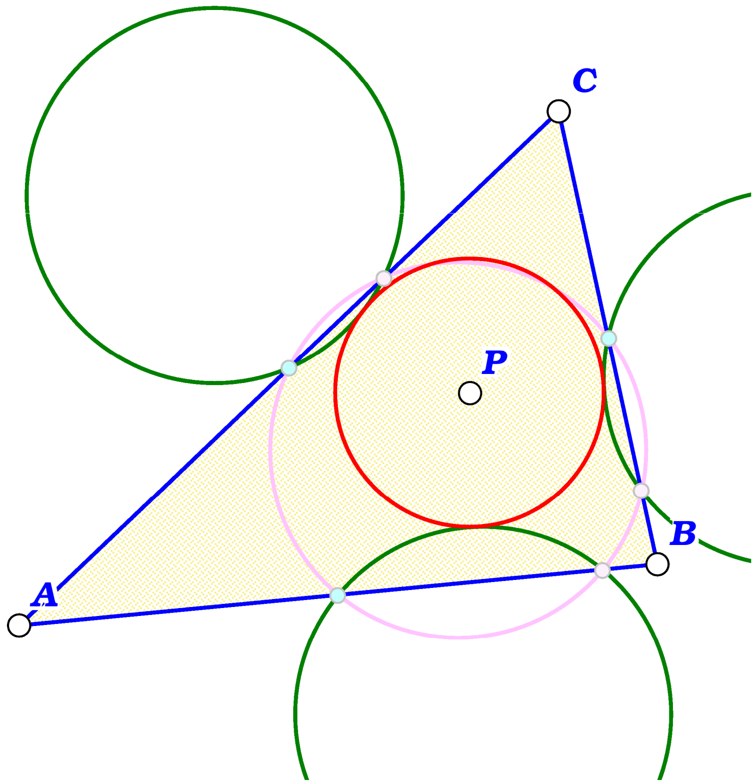
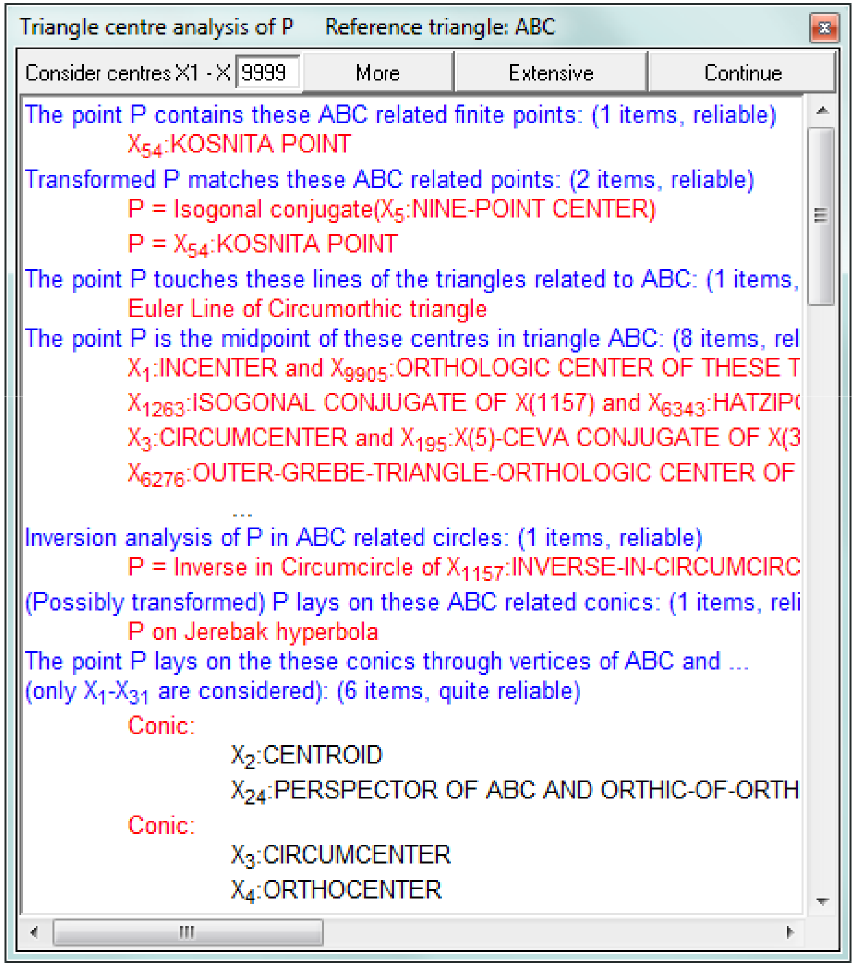
The 9-point circle of a triangle is mirrored in each triangle's side.



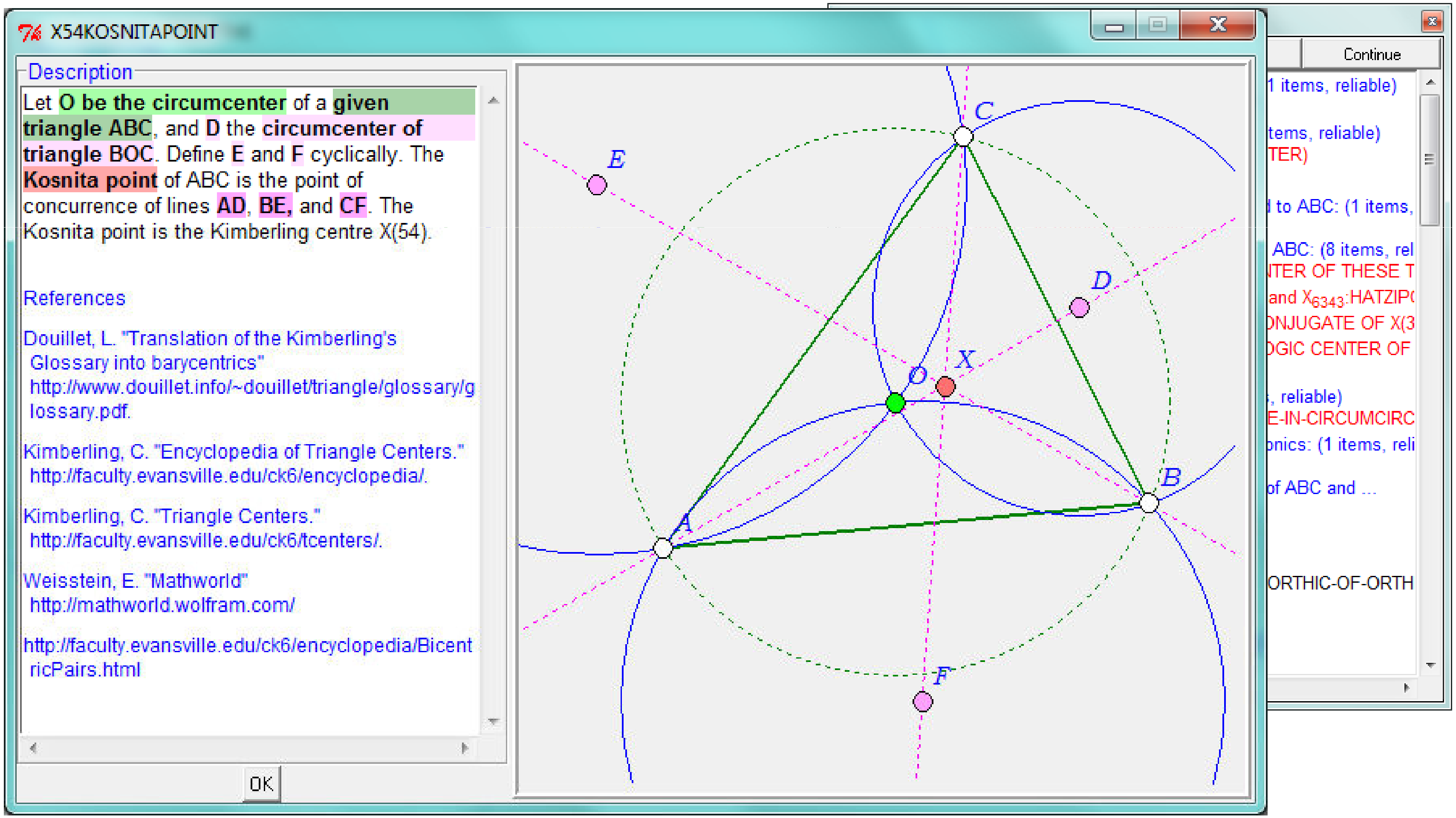


A circle is fits between the mirrored circles. What point is its centre P?

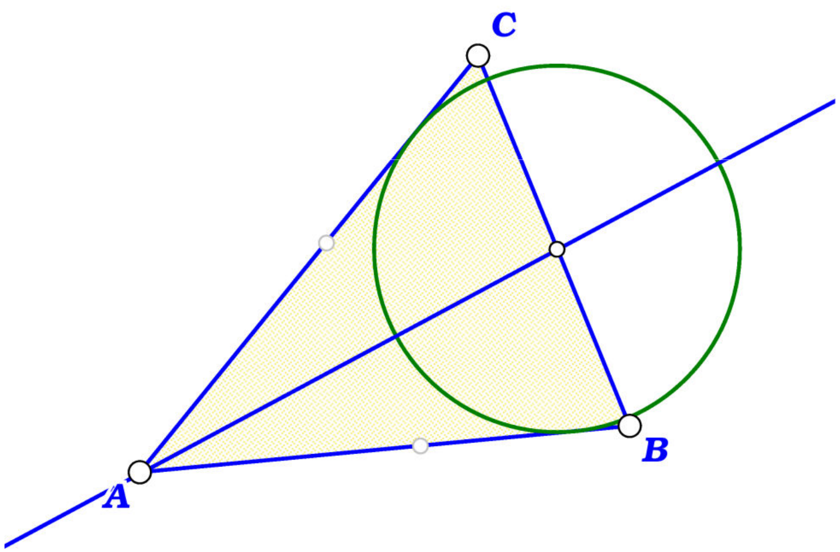




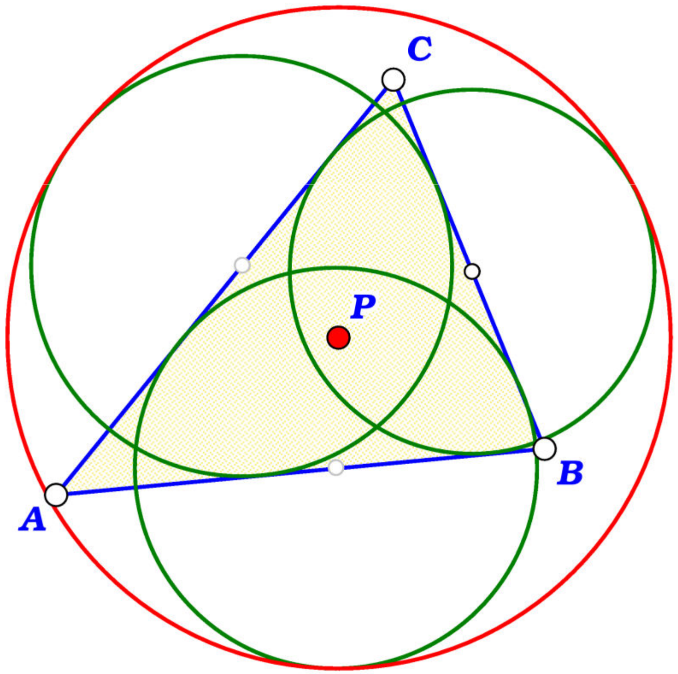
Additional explanation of P

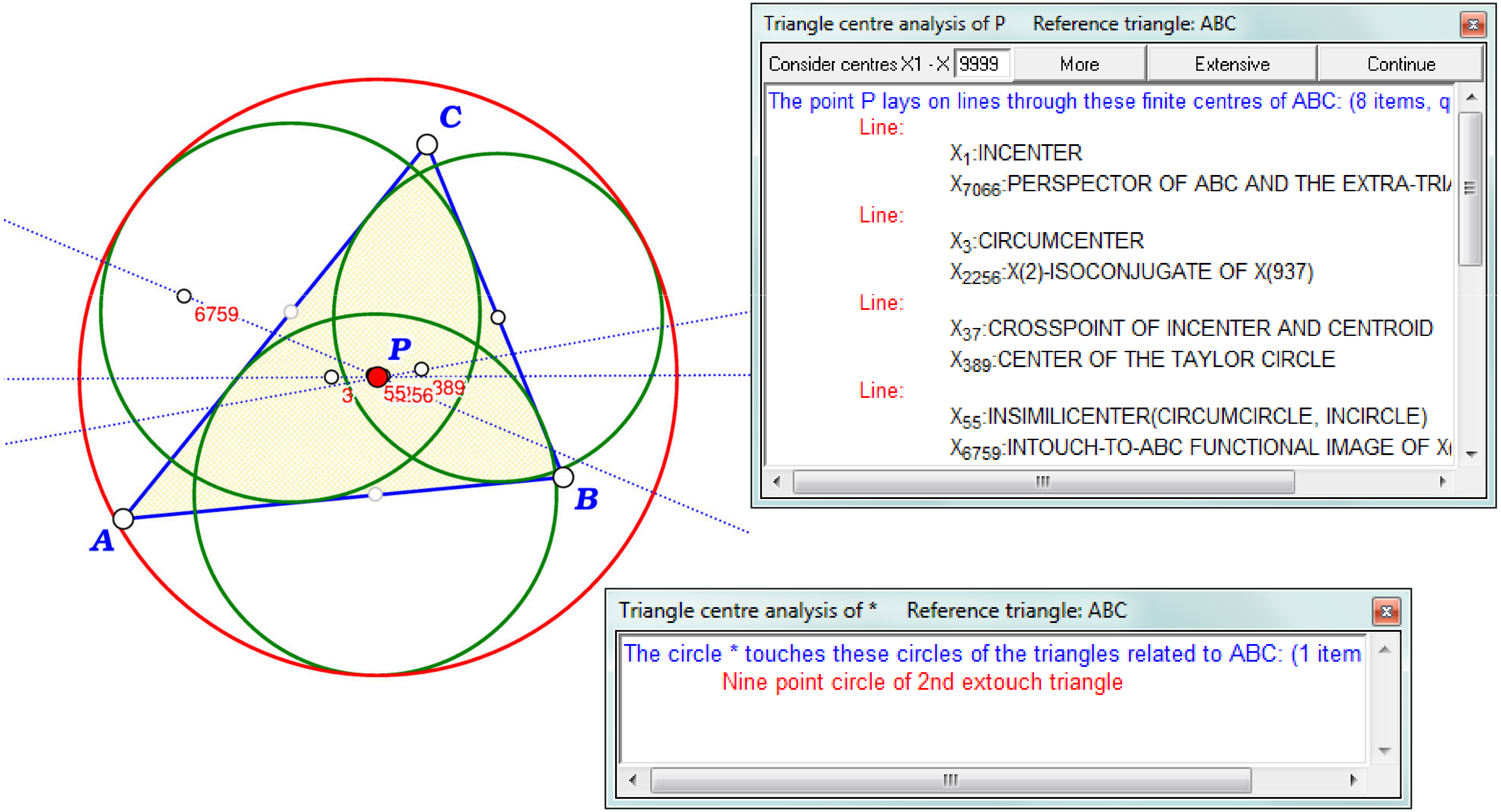


In a triangle ABC consider the circles with center on one side and touching the other two sides.

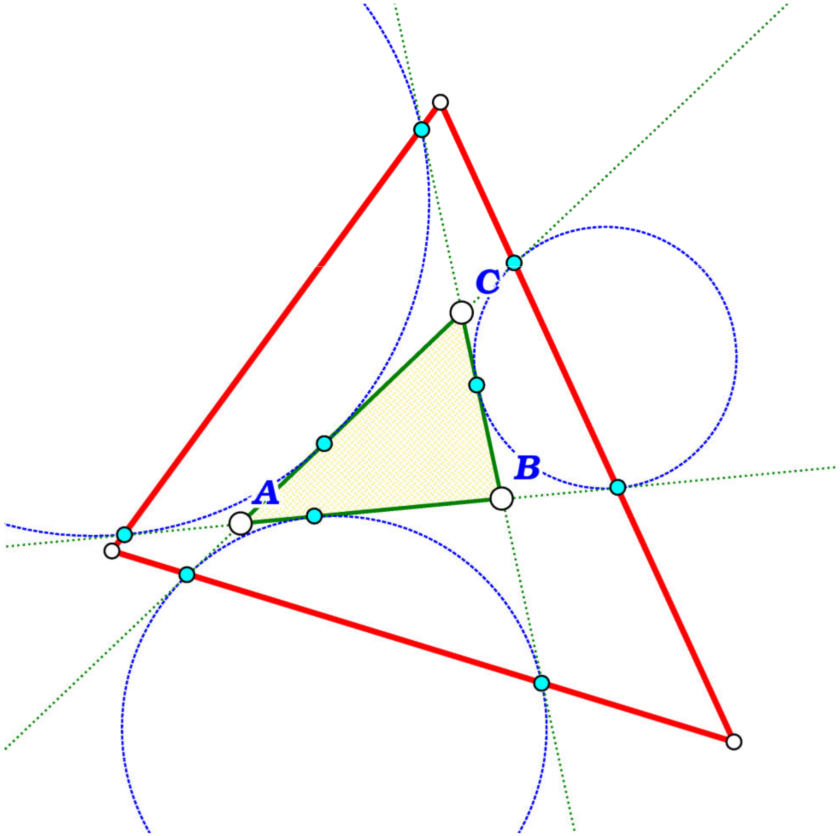


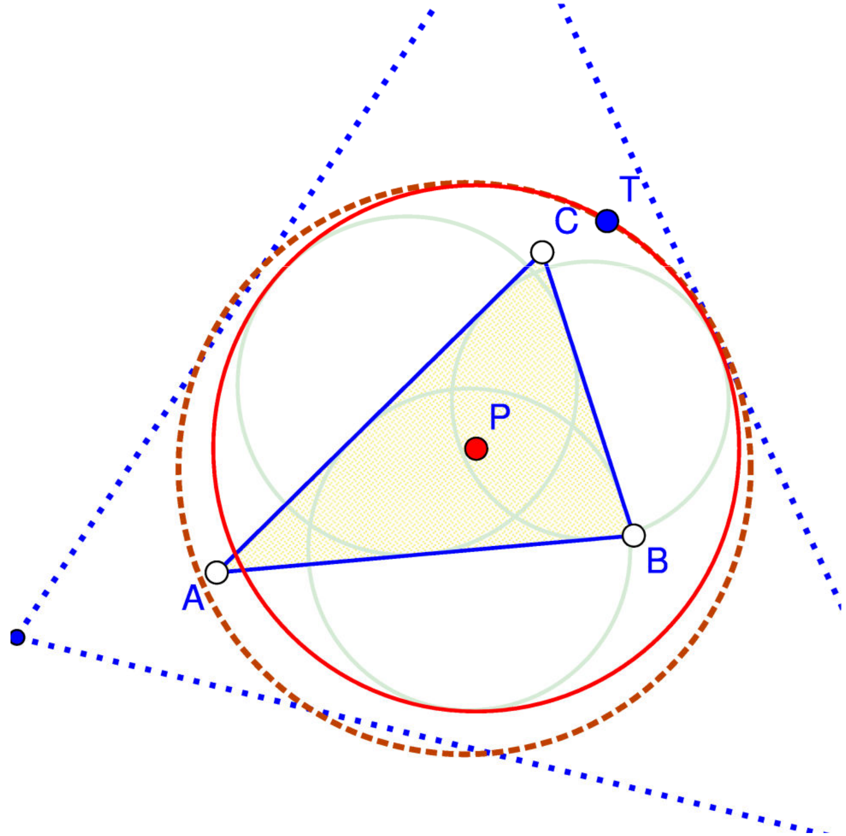
Consider the circle touching the three circles from outside. What can be said about this circle and its center?




C
6759
P
3
55
56
389
A
B
Triangle centre analysis of P Reference triangle: ABC
Consider centres X1 - X 9999
More
Extensive
Continue
The point P lays on lines through these finite centres of ABC: (8 items, q
Line:
X1:INCENTER
X7066:PERSPECTOR OF ABC AND THE EXTRA-TRI
Line:
X3:CIRCUMCENTER
X2256:X(2)-ISOCONJUGATE OF X(937)
Line:
X37:CROSSPOINT OF INCENTER AND CENTROID
X389:CENTER OF THE TAYLOR CIRCLE
Line:
X55:INSIMILICENTER(CIRCUMCIRCLE, INCIRCLE)
X6759:INTOUCH-TO-ABC FUNCTIONAL IMAGE OF X
Triangle centre analysis of * Reference triangle: ABC
The circle * touches these circles of the triangles related to ABC: (1 item
Nine point circle of 2nd extouch triangle

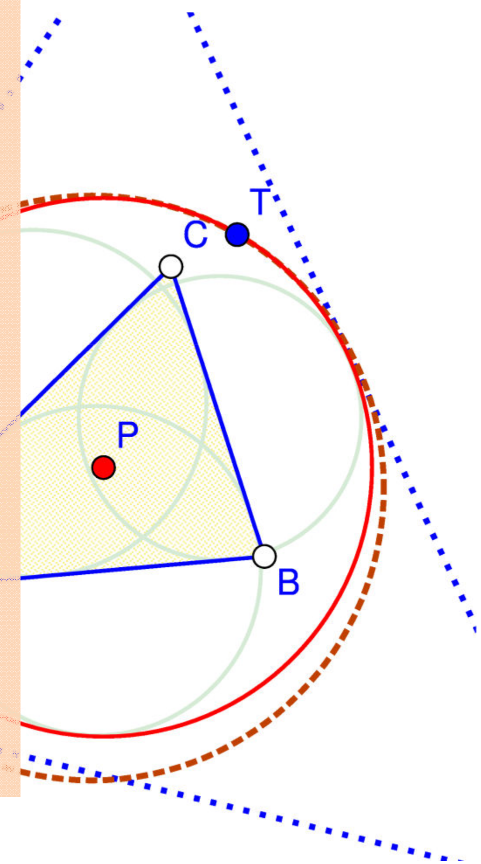
2nd excentral triangle





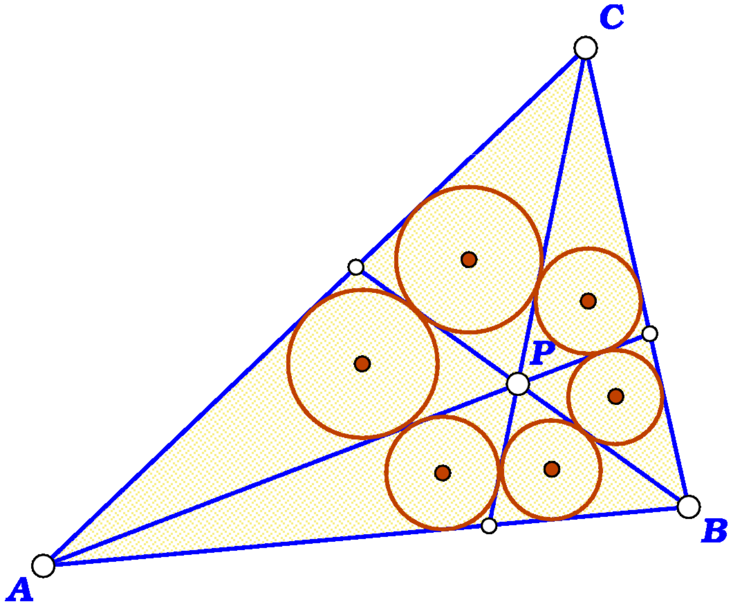
In the triangle observation OK Geometry considers

- 10 000 triangle centres (-> geometric meanings)
- Millions of lines through these centres
- ~50 geometric operations among the centres
- ~400 conics related to the reference triangle
- ~90 triangles related to the reference triangle
- ~10 lines related to each of these triangles
- ~30 circles related to each of these triangles

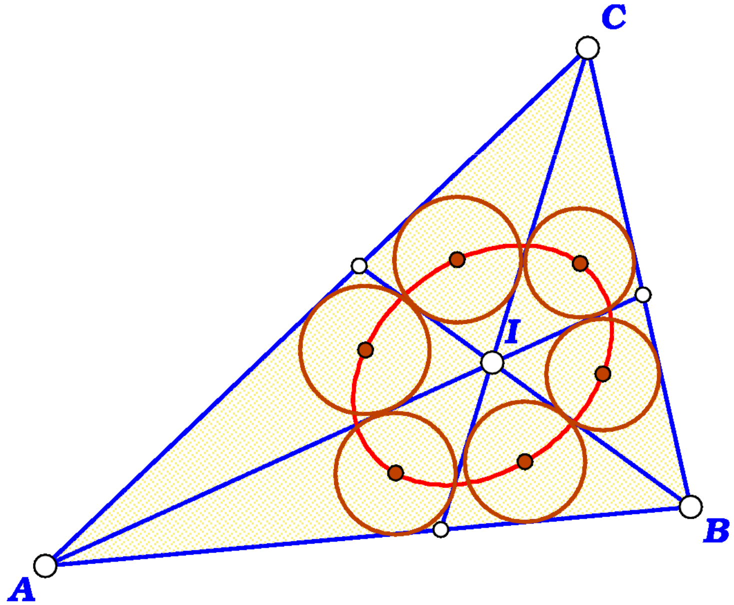


2nd excentral triangle

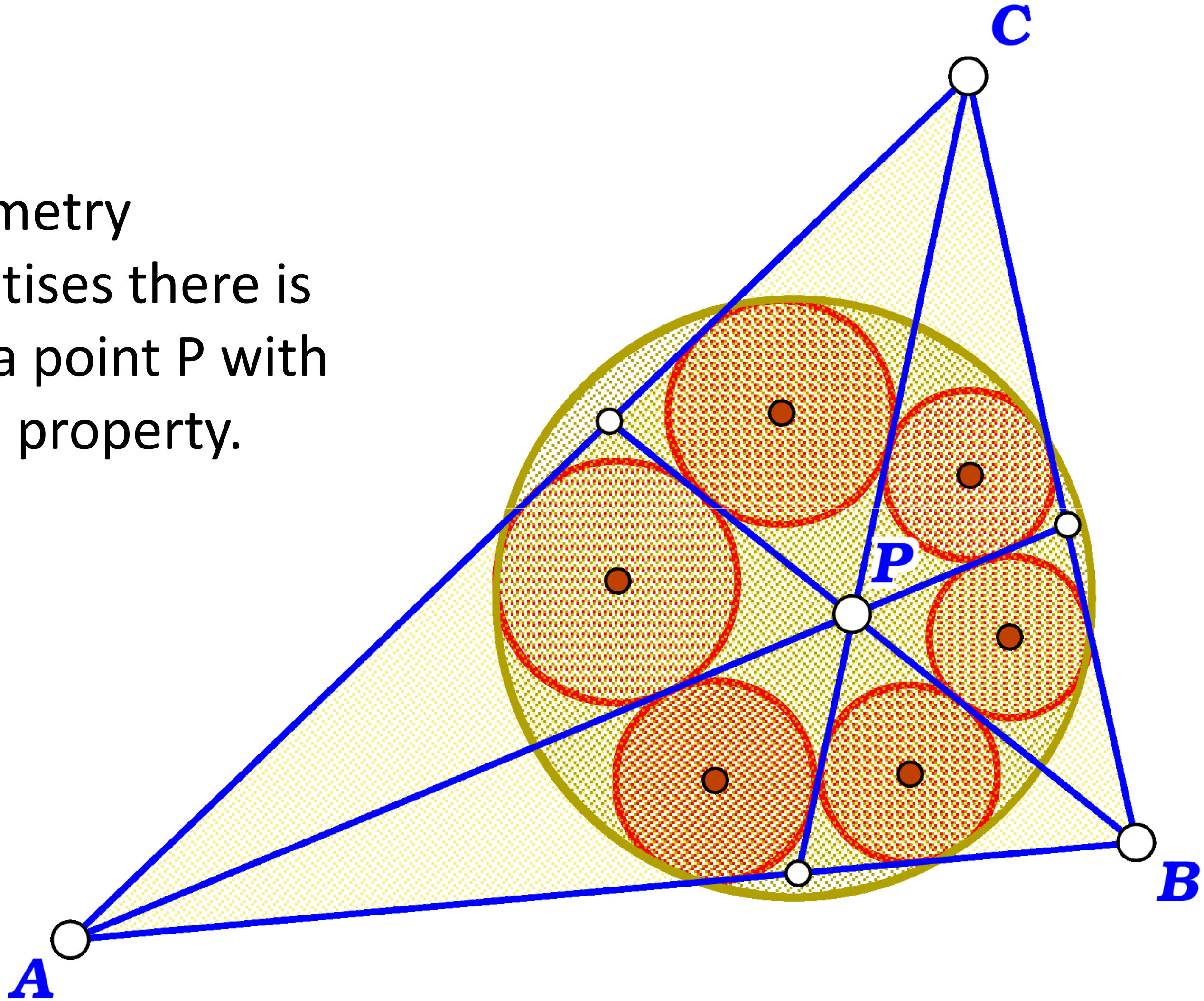
A point P in a triangle gives rise to 6 small triangles. The incentres of these triangles, in general, do not lay on a same conic.



The incentres lay on the same conic if, for example, P is the incentre of ABC.



OK Geometry hypothetises there is allways a point P with this nice property.



Technology and writing up proofs

- Various dimensions of comprehending proofs (Weber, Selden,…)
- Various levels of comprehending proofs (Lin)
- Various presentation modes (Herbst, Wong,…)
- Various specific tools

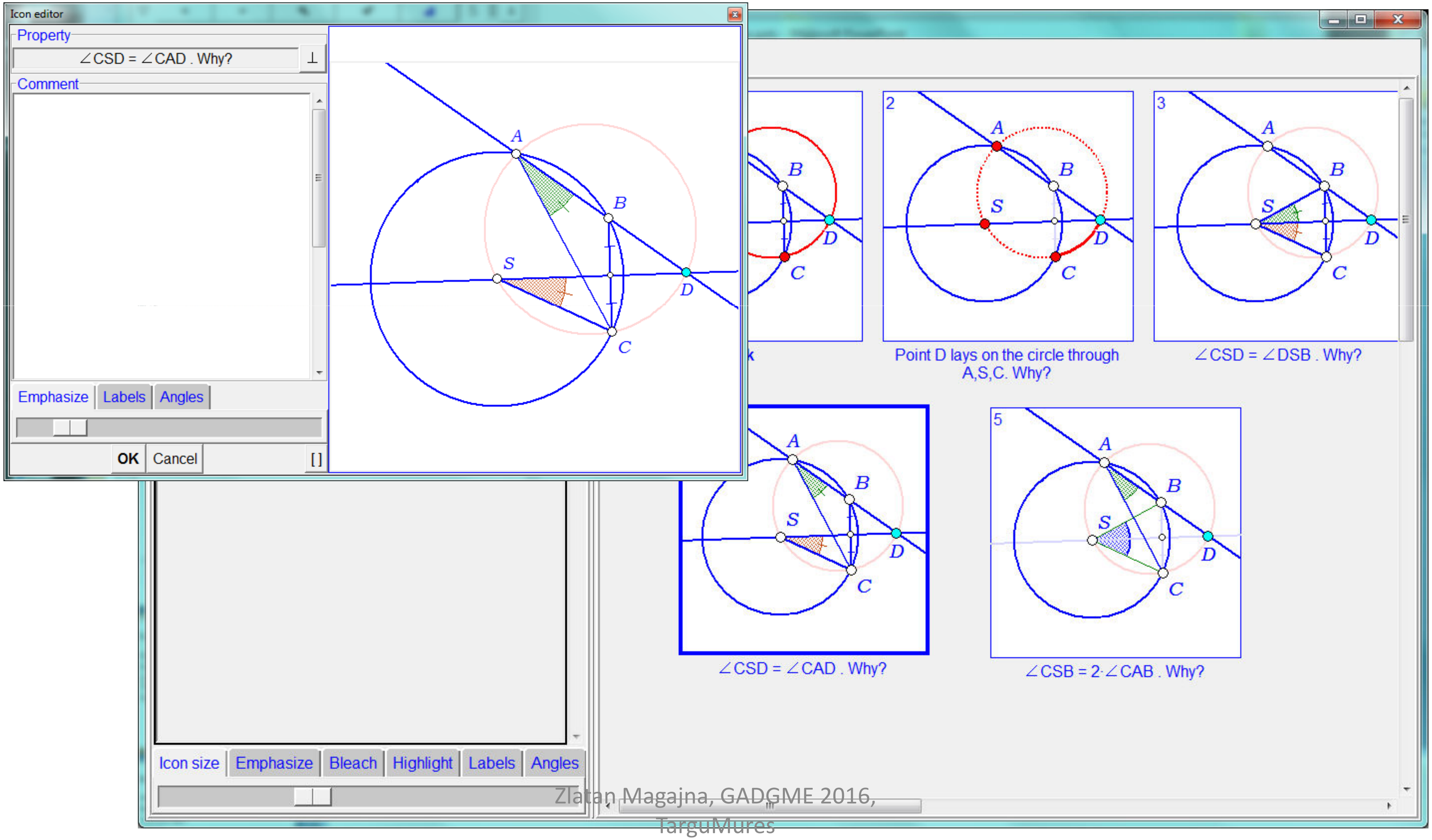
- There is not a single good way of presenting a proof.

Technology and writing up proofs

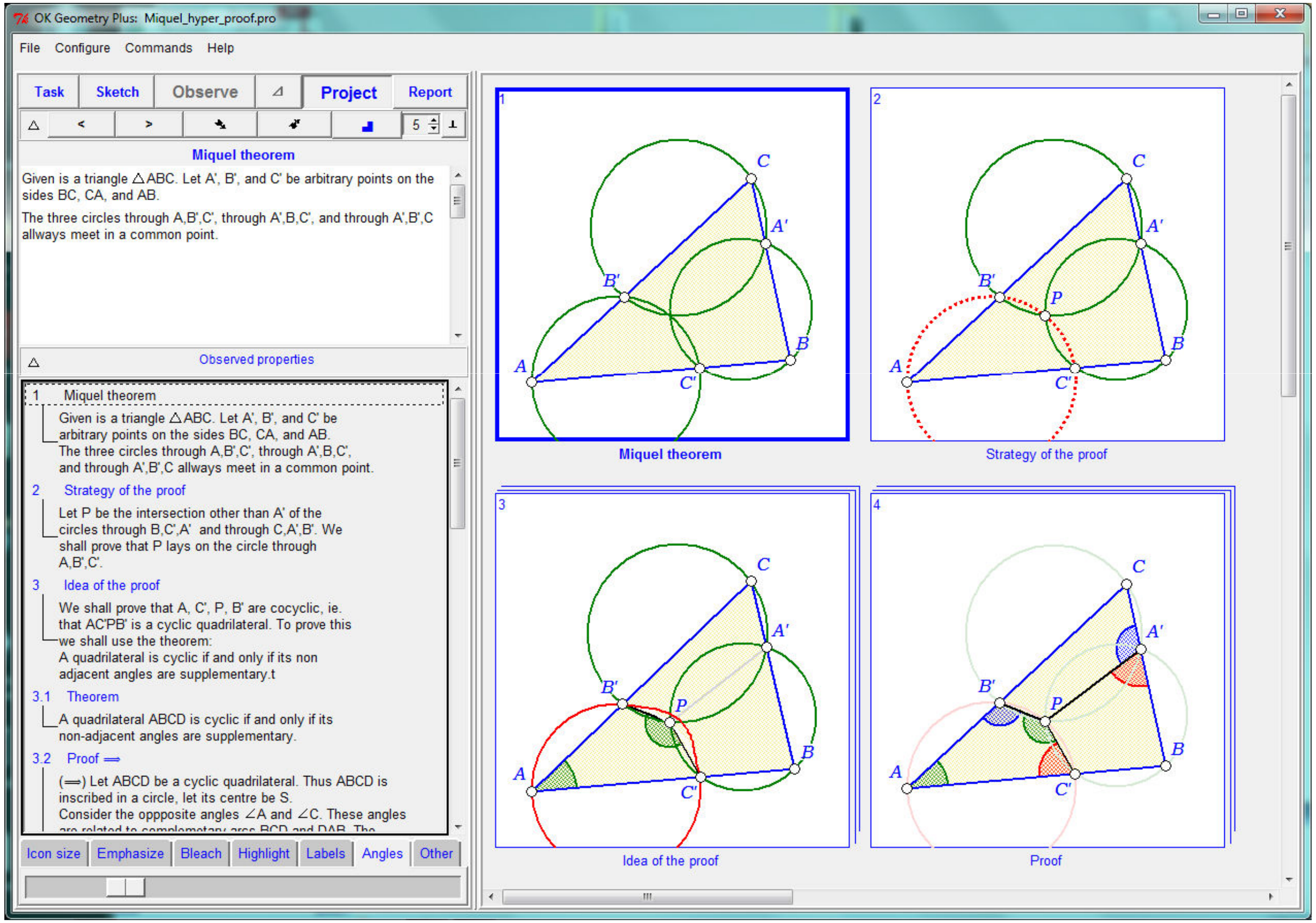
- Various dimensions of comprehending proofs (Weber, Selden,…)
- Various levels of comprehending proofs (Lin)
- Various presentation modes (Herbst, Wong,…)
- Various specific tools

- Meaning of terms and statements
- Justification of claims
- Logical structure (proof framework)
- High level ideas (structure of proof)
- General method
- Application of proof

Justification of claims



High-level ideas



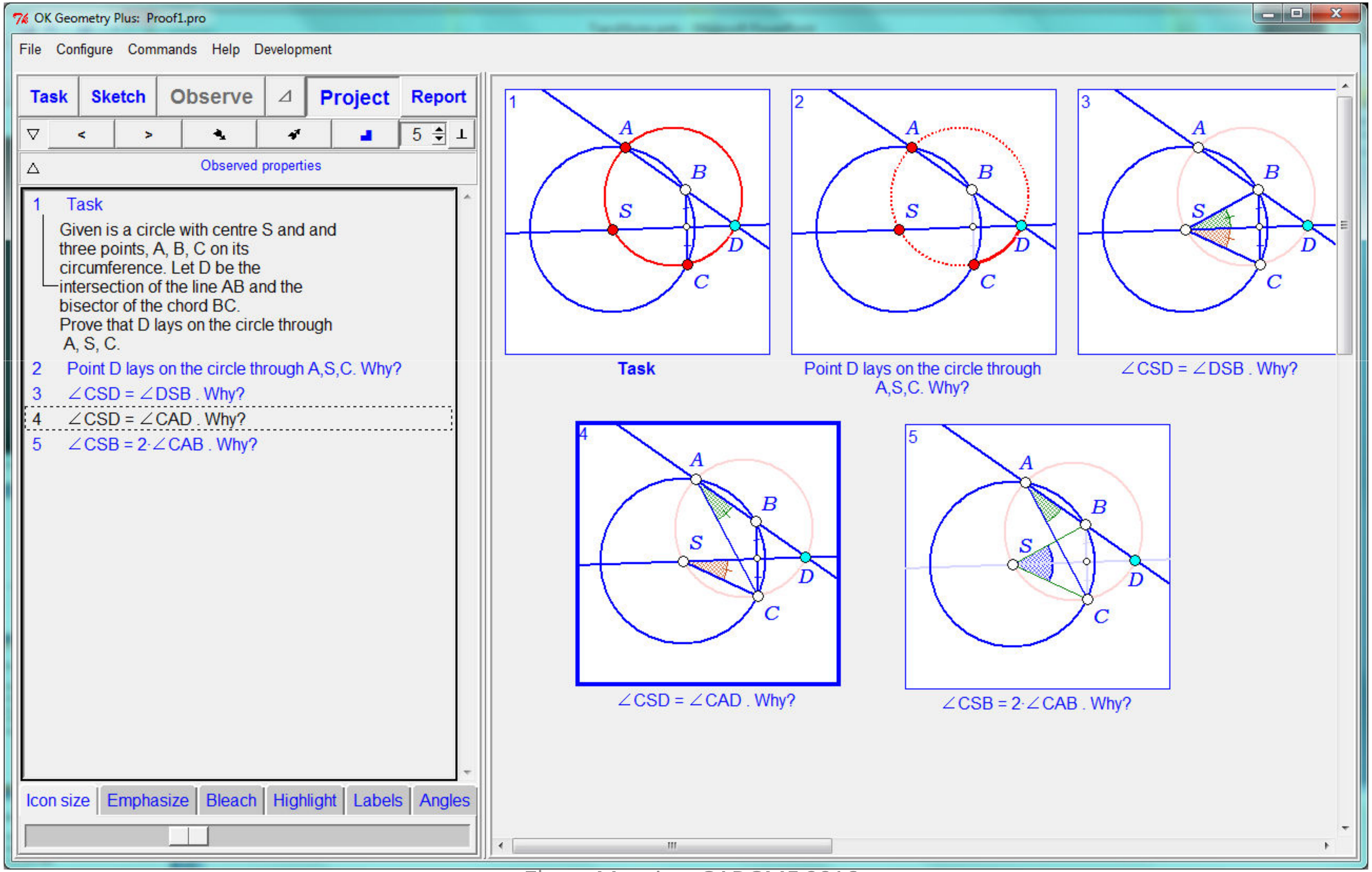
Technology and writing up proofs

- Various dimensions of comprehending proofs (Weber, Selden,…)
- Various levels of comprehending proofs (Yang, Lin)
- Various presentation modes (Herbst, Wong,…)
- Various specific tools

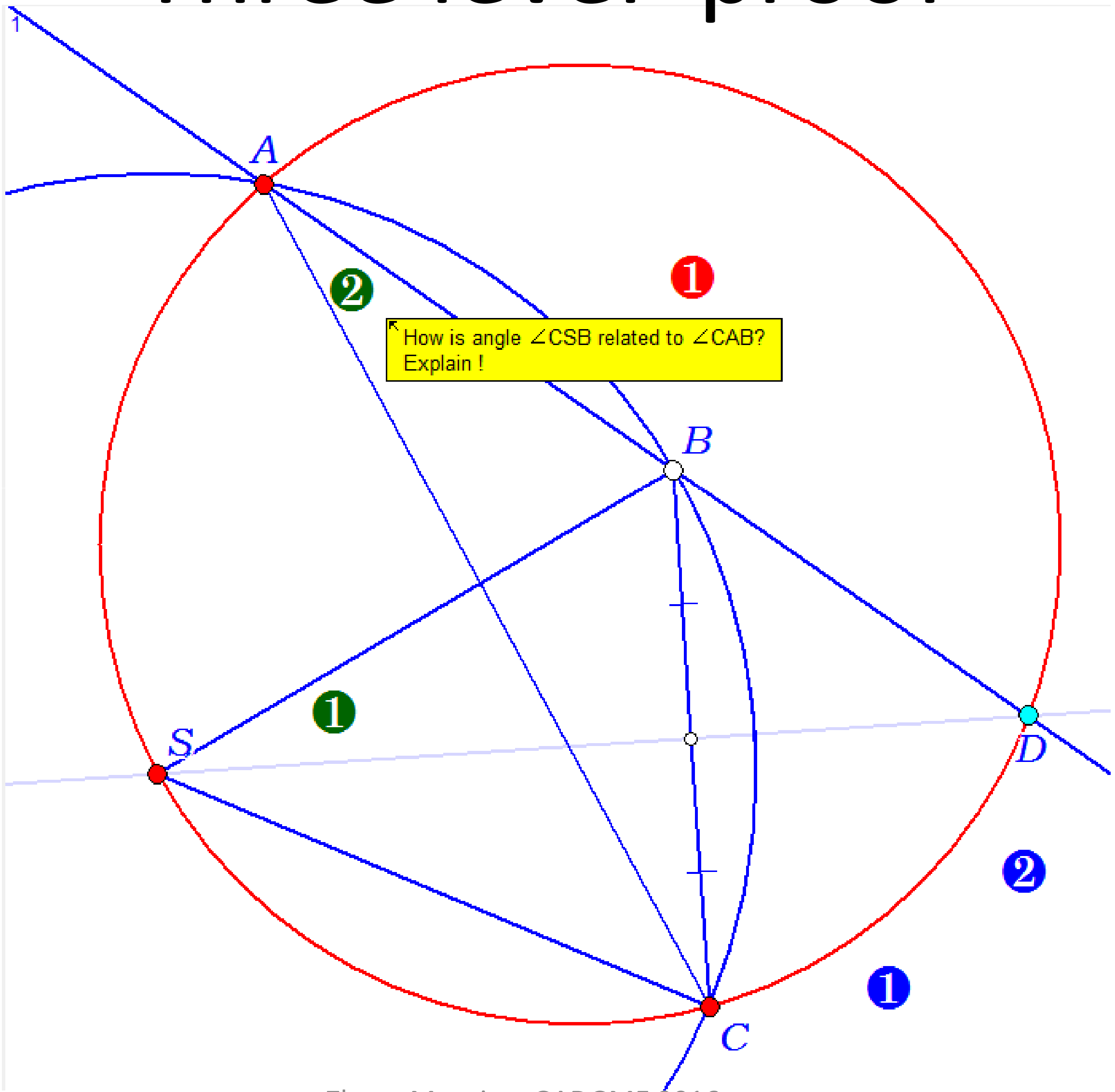
- Surface
- Recognising elements
- Chaining elements
- Encapsulation

(Scaffolding at various levels)

Chaining elements



‘Three level’ proof



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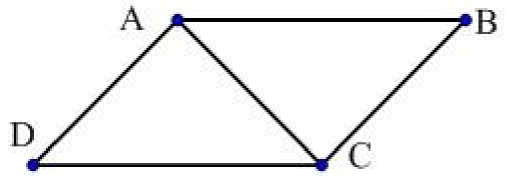
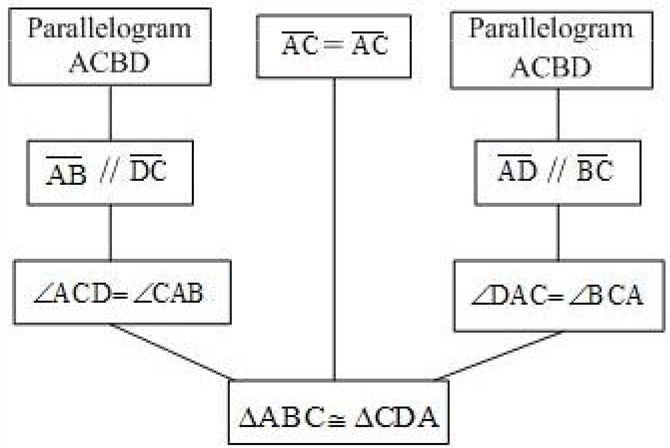
- Figure, dynamic figure
- Flowchart
- Two column mode
- Paragraph mode

Multiple representations – Mr Geo
(Wong, Yin, Yang, Cheng, 2011)

Given: Parallelogram ABCD
with diagonal $\overline{AC}$
Prove: $\Delta ABC \cong \Delta CDA$



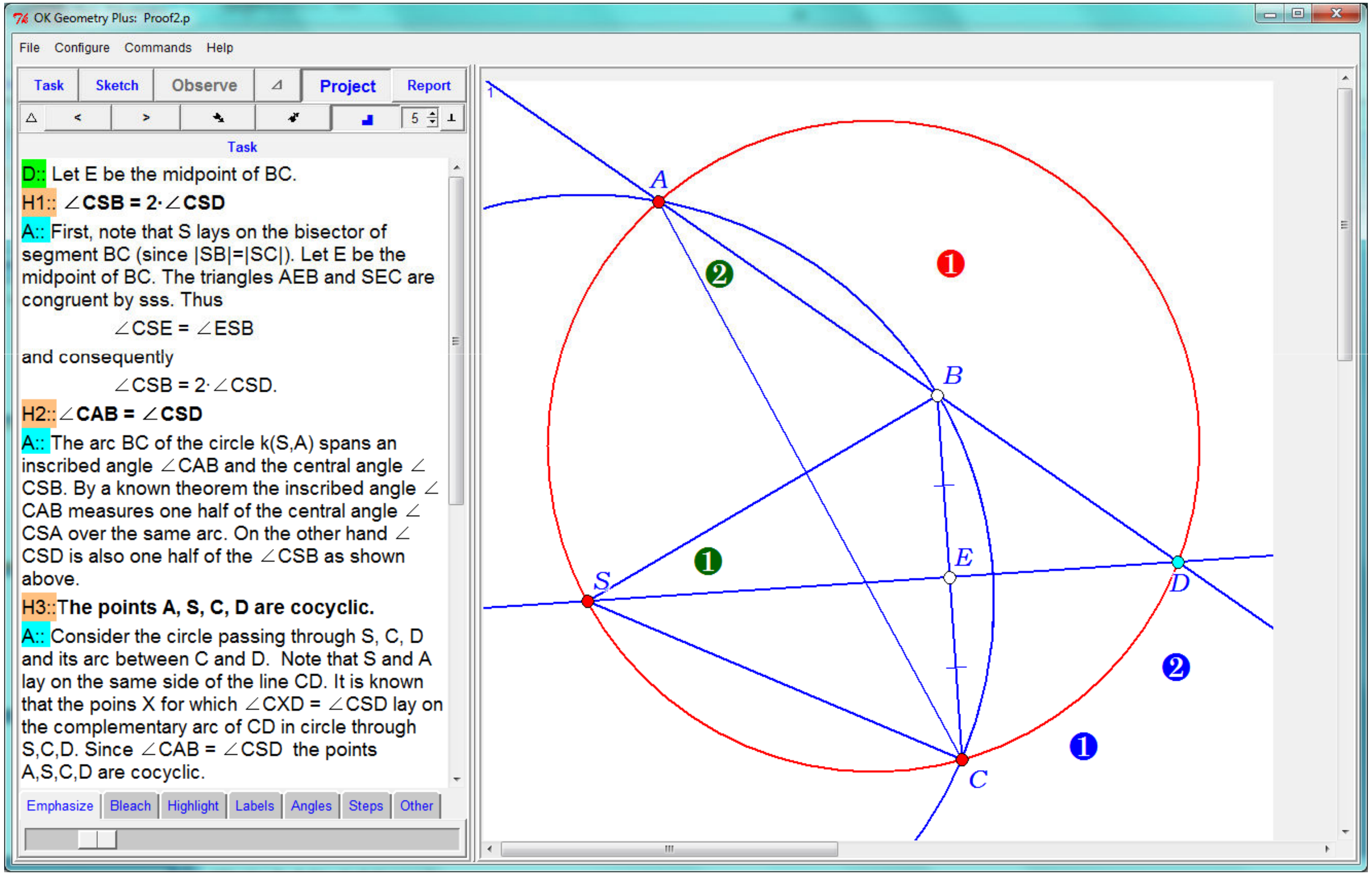
1. ABCD is a parallelogram (Given)
2. $\because$ ABCD is a parallelogram, $\therefore$ $\overline{AB} \mathbin{//} \overline{DC}$
(Def. of parallelogram)
3. $\because$ ABCD is a parallelogram, $\therefore$ $\overline{AD} \mathbin{//} \overline{BC}$
(Def. of parallelogram)
4. $\because \overline{AB} \mathbin{//} \overline{DC}$, $\therefore \angle ACD = \angle CAB$ (Alt. int. anlges)
5. $\because \overline{AD} \mathbin{//} \overline{BC}$, $\therefore \angle DAC = \angle BCA$ (Alt. int. anlges)
6. $\overline{AC} = \overline{AC}$ (Reflexive law)
7. $\Delta ABC \cong \Delta CDA$ (ASA)



Technology and writing up proofs

- Various dimensions of comprehending proofs (Weber, Selden,...)
- Various levels of comprehending proofs (Lin)
- Various presentation modes (Herbst, Wong,...)
- Various specific tools

A D H statements



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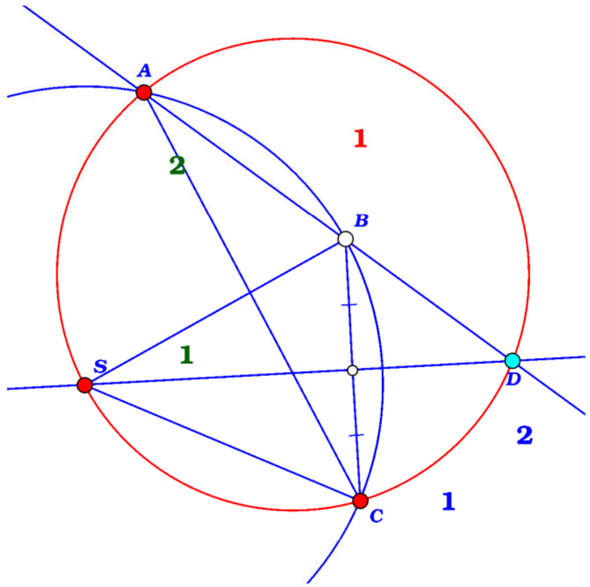


Task

1 Task

Given is a circle with centre S and and three points, A, B, C on its circumference. Let D be the intersection of the line AB and the bisector of the chord BC.

Prove that S, C, D, and A are cocyclic.

Comment:

2 Proof

Definition Let E be the midpoint of BC.

Claim 1 ∠CSB = 2·∠CSD

Argument 1 First, note that S lays on the bisector of segment BC (since |SB|=|SC|). Let E be the midpoint of BC. The triangles AEB and SEC are congruent by sss. Thus

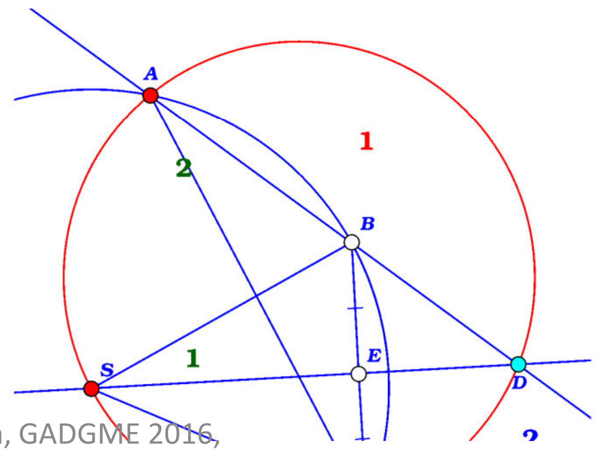
∠CSE = ∠ESB

and consequently

∠CSB = 2·∠CSD.

Claim 2 ∠CAB = ∠CSD

Argument 2 The arc BC of the circle k(S,A) spans an inscribed angle ∠CAB and the central angle ∠CSB. By a k



Thank you!

OK Geometry

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